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Analytical Description of the Volume, Center of Gravity, and Moments of Inertia for Tangent and Secant Ogives

by Steven B Segletes

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14. ABSTRACT
This note presents closed-form algebraic equations to describe the inertial properties of both tangent ogives, which conform to the cylindrical shank of the projectile in a flush manner, and secant ogives, which are base-truncated versions of tangent ogives. The properties encompassed by these descriptions include volume (mass), center of gravity, and the various moments of inertia.

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1. Introduction

The inertial properties* of regularly shaped bodies are required for studies in rigid-body dynamics. Tabulated lists of these properties for a variety of different geometries exist in the textbook literature.[1] However, the ballistic ogive, often appearing at the front of a cylindrical projectile, is one such shape that appears to have eluded study, in this regard. In this note, we remedy that deficiency.

2. The Ballistic Ogive

A ballistic ogive is a solid of revolution whose rear surface is planar and whose front surface is formed by rotating a convex circular arc about the symmetry axis. The ballistic ogive may be characterized, in general, by three dimensions: the caliber radius head (CRH), the ballistic length (BL), and the projectile diameter D.[2,3]

The CRH is a nondimensional number defined as

$$\mathsf{CRH} = \frac{R_{\text{ogive}}}{D} \quad ,$$

where R_{ogive} is the radius of curvature of the circular arc defining the ogival head. The BL of a projectile is the length of its ogival head measured along the projectile's symmetry axis. An ogive schematic is given, for the general case, in Fig. 1.

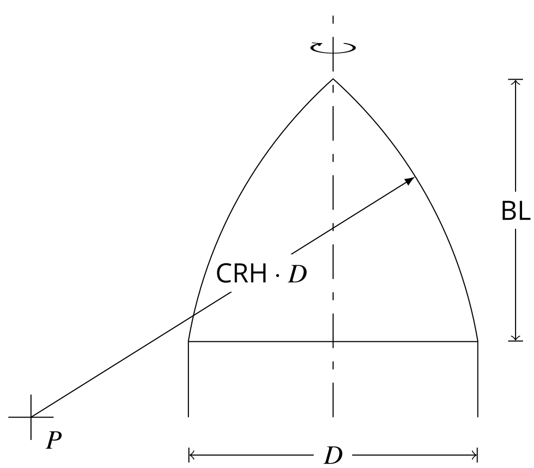

Fig. 1 The parameters describing an ogive: CRH, BL, and D. In the general case, the center of radius P may lie below the basal plane, in which case the body is referred to as a secant ogive.

*By inertial properties, we refer to the volume (or mass), center of gravity (CG), and the three respective moments of inertia (though, in the z-axisymmetric case, $I_{xx} = I_{yy}$).

However, we begin this note by limiting our consideration to the tangent ogive, for which the rearward end of the ogive blends smoothly into the shank of the cylindrical projectile that follows, without a slope dislocation known as a “shoulder”. Said differently, a tangent ogive is one where the center of the arc that traces the ogival nose (denoted as P in Fig. 1) lies in the basal plane of the ogive (the plane $z = 0$, as in Fig. 2). Such a condition ensures a continuous slope at the point where the ogive joins the cylindrical body that follows. We see in the following section that, for tangent ogives, $\mathsf{BL}_{\mathrm{tan}}$ is not a free parameter, but may be expressed in terms of the CRH and projectile caliber.

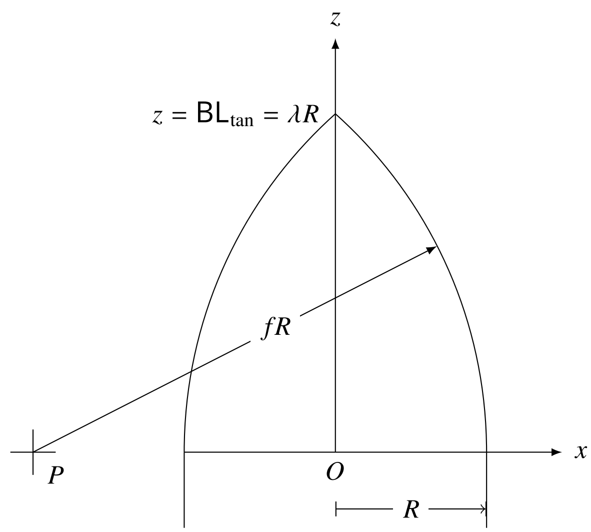

Fig. 2 The parameters used in this report to describe a tangent ogive: f and R. For a tangent ogive, the center of radius P must lie in the plane $z = 0$.

3. Parameters Governing the Geometry of the Tangent Ogive

It is convenient to introduce the parameter f, defining the ogival-head radius of curvature (see Fig. 2), nondimensionalized by the projectile radius R:

$$f = \frac{R_{\mathrm{ogive}}}{R} \quad .$$

In Fig. 2, the ballistic length $\mathsf{BL}_{\mathrm{tan}}$ for a tangent ogive, nondimensionalized by the projectile radius R, is designated as λ. Consider sweeping the fR ray, anchored at P, counterclockwise until it connects to the tip of the ogive. In this case, a right triangle is formed by the origin O, center-of-arc P, and the ogive tip (side lengths

$(f-1)R$, fR, and λR). Application of the Pythagorean Theorem leads to

$$f^2 = (f-1)^2 + \lambda^2 \quad ,$$

through which λ may be solved:

$$\lambda = \frac{\mathsf{BL}_{\tan}}{R} = \sqrt{2f-1} \quad . \tag{1}$$

Equation 1 reveals that, for tangent ogives, the nondimensional ballistic length λ is not a free parameter, but is constrained by the value of f. For tangent ogives, the CRH, $\mathsf{BL}_{\tan}$, and the caliber may all be expressed solely in terms of the two parameters, f and R:

$$\begin{aligned} \mathsf{CRH} &= f/2 \\ \mathsf{BL}_{\tan} &= \sqrt{2f-1}R \quad . \\ D &= 2R \end{aligned}$$

As a point of interest, there exists only one nondimensional, tangent-ogival geometry for which the nondimensional ballistic length λ equals the CRH, namely, the case where CRH $= 2 + \sqrt{3} \approx 3.732$. In contrast, a value of CRH $= 0.5$ corresponds to a hemispherical ogive. In all cases, $f \geq 1$ is a geometrical constraint.

4. Inertial Properties of the Tangent Ogive

Using direct integration with standard tables,[4] the inertial properties of ogives may be ascertained. Because the integration process is tedious, only the final nondimensional results are presented here. However, Appendix A may be consulted for the particulars of the various integrations.

The inertial properties of the tangent ogive expressed are normalized by way of powers of R and expressed in terms of f and λ. Note, however, that in light of Eq. 1, λ itself is defined in terms of f. Therefore, the mere specification of the nondimensional f term (and material density ρ) uniquely defines the nondimensional inertial properties of the tangent ogive.

The ogival volume V may be calculated as

$$\frac{V}{R^3} = \pi\left(\left(f^2 - \frac{1}{3}\lambda^2\right)\lambda - f^2(f-1)\sin^{-1}\left(\frac{\lambda}{f}\right)\right) \quad . \tag{2}$$

The mass of the ogive, m, may be expressed in terms of Eq. 2 as

$$\frac{m}{R^3} = \rho \frac{V}{R^3} \quad ,$$

where ρ is the density of the ogive material.

The CG of the tangent ogive lies along the axis of symmetry at a distance $\bar{z}$ from the ogival base, given by

$$\frac{\bar{z}}{R} = \frac{\pi}{V/R^3}\left(-\frac{2}{3}(f-1)\big(f^3-(f-1)^3\big)+\frac{1}{2}\big(f^2+(f-1)^2\big)\lambda^2-\frac{1}{4}\lambda^4\right) \quad . \quad (3)$$

Note that V/R^3, appearing in the denominator, is available from Eq. 2.

The moment of inertia about the axis of symmetry of the tangent ogive is given as

$$\begin{aligned}\frac{I_{zz}}{\rho R^5} = \frac{\pi}{2}\Bigg(&\left(f^4+\frac{9}{2}f(f-1)^2-2(f-1)^4\right)\lambda-\left(\frac{2}{3}f^2+2(f-1)^2\right)\lambda^3+\frac{1}{5}\lambda^5 \\ &-f^2(f-1)\left(\frac{3}{2}f^2+2(f-1)^2\right)\sin^{-1}\left(\frac{\lambda}{f}\right)\Bigg)\end{aligned} \quad .(4)$$

An alternate (but equivalent) expression may also be derived for I_{zz}, by using the shell method of integration (rather than the disk method). Because it requires a recursive integration to reduce the order of the integrand, it yields an integral in which the integration constants b_i are recursively calculated:

$$\frac{I_{zz}}{\rho R^5} = \pi\left(\left(f^2\sin^{-1}\left(\frac{\lambda}{f}\right)-(f-1)\lambda\right)b_0-2\left(\frac{b_1}{3}+\frac{b_2}{4}+\frac{b_3}{5}\right)\lambda^3\right) \quad , \qquad (5)$$

where

$$\begin{aligned} b_3 &= -1 \\ b_2 &= +3 + \tfrac{7}{5}\, b_3 f \\ b_1 &= -3 + \tfrac{5}{4}\, b_2 f \\ b_0 &= +1 + \tfrac{3}{3}\, b_1 f \end{aligned} \quad .$$

The moment of inertia of the tangent ogive, about an axis lying in the *basal plane* of the ogive and perpendicular to the symmetry axis, is

$$\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{base}} = \left.\frac{I_{yy}}{\rho R^5}\right|_{\text{base}} = \frac{\pi}{4}\Bigg(f^2\left(f^2+\frac{7}{2}(f-1)^2\right)\lambda+\frac{1}{15}\lambda^5 - f^2(f-1)\left(\frac{5}{2}f^2+2(f-1)^2\right)\sin^{-1}\left(\frac{\lambda}{f}\right)\Bigg) \quad . \tag{6}$$

The comparable moment of inertia *about the CG* follows directly from the parallel-axis theorem, such that

$$\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{cg}} = \left.\frac{I_{xx}}{\rho R^5}\right|_{\text{base}} - \frac{V}{R^3}\left(\frac{\bar{z}}{R}\right)^2 \tag{7}$$

and likewise for $I_{yy}\big|_{\text{cg}}$.

One may express the previous moments of inertia in terms of the ogive mass using Eq. 2:

$$\frac{I_{ii}}{mR^2} = \frac{I_{ii}}{\rho R^5}\cdot\frac{1}{V/R^3} \quad , \tag{8}$$

where $ii = xx, yy$, or zz.

4.1 Special-Case Check: Hemispherical Ogive

The prior relations may be checked against known special cases for which analytical results also exist. One such case is the hemispherical ogive, in which $f = \lambda = 1$. For this special case, it may be verified with the formulae (Eqs. 2–8) that the volume and CG of the hemispherical ogive are, respectively,

$$\frac{V}{R^3} = \frac{2\pi}{3} \quad ,$$

$$\frac{\bar{z}}{R} = \frac{3}{8} \quad .$$

The moments of inertia evaluate as

$$\frac{I_{zz}}{\rho R^5} = \frac{4\pi}{15} \quad ,$$

$$\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{base}} = \left.\frac{I_{yy}}{\rho R^5}\right|_{\text{base}} = \frac{4\pi}{15} \quad ,$$

$$\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{cg}} = \left.\frac{I_{yy}}{\rho R^5}\right|_{\text{cg}} = \frac{83\pi}{480} \quad .$$

In terms of mass, the moments of inertia are

$$\frac{I_{zz}}{mR^2} = \frac{2}{5} \quad ,$$

$$\left.\frac{I_{xx}}{mR^2}\right|_{\text{base}} = \left.\frac{I_{yy}}{mR^2}\right|_{\text{base}} = \frac{2}{5} \quad ,$$

$$\left.\frac{I_{xx}}{mR^2}\right|_{\text{cg}} = \left.\frac{I_{yy}}{mR^2}\right|_{\text{cg}} = \frac{83}{320} \quad .$$

These results are wholly in agreement with published results for hemispherical bodies.[5]

4.2 General Check: Numerical Integration

The derived inertial properties for tangent ogives are also validated against the comparable numerical integrations (using finite disks and shells). The *f77* code presented in Appendix B was exercised for purposes of validating the analytical results derived in this note.

In the numerically integrated comparison, shown in Table 1 for four different cases of ogive radius, the ogival domain was successively discretized into either 1000 stacked disks or 1000 concentric shells in order to numerically evaluate the inertial properties. Validation is achieved, though it is worthy to note that several orders-of-magnitude finer discretization is needed for the numerical integration to match the analytical results to 5 decimal places shown in the table, especially as f grows larger.

Table 1 Validation of analytically derived inertial properties for tangent ogives, for various values of nondimensionalized radius-of-ogive-curvature, f

Case: $f = 1$ ($\lambda = 1.000$)

	Analytical	Numerical
V/R^3	2.09440	2.09441
$\bar{z}/R$	0.37500	0.37566
$\frac{I_{zz}}{\rho R^5}$	0.83776	0.83778
$\left.\frac{I_{xx}}{\rho R^5}\right\|_{\text{base}}$	0.83776	0.83779
$\left.\frac{I_{xx}}{\rho R^5}\right\|_{\text{cg}}$	0.54323	0.54222

Case: $f = 2$ ($\lambda = 1.732$)

	Analytical	Numerical
V/R^3	3.16472	3.16475
$\bar{z}/R$	0.57907	0.57995
$\frac{I_{zz}}{\rho R^5}$	1.23195	1.23198
$\left.\frac{I_{xx}}{\rho R^5}\right\|_{\text{base}}$	2.14828	2.14840
$\left.\frac{I_{xx}}{\rho R^5}\right\|_{\text{cg}}$	1.08708	1.08398

Case: $f = 6$ ($\lambda = 3.317$)

	Analytical	Numerical
V/R^3	5.69928	5.69932
$\bar{z}/R$	1.05652	1.05798
$\frac{I_{zz}}{\rho R^5}$	2.18578	2.18582
$\left.\frac{I_{xx}}{\rho R^5}\right\|_{\text{base}}$	10.36115	10.36193
$\left.\frac{I_{xx}}{\rho R^5}\right\|_{\text{cg}}$	3.99946	3.98261

Case: $f = 10$ ($\lambda = 4.3589$)

	Analytical	Numerical
V/R^3	7.41229	7.41234
$\bar{z}/R$	1.37747	1.37934
$\frac{I_{zz}}{\rho R^5}$	2.83497	2.83502
$\left.\frac{I_{xx}}{\rho R^5}\right\|_{\text{base}}$	21.94154	21.94327
$\left.\frac{I_{xx}}{\rho R^5}\right\|_{\text{cg}}$	7.87736	7.84080

5. Inertial Properties of the Secant Ogive

Mathematically, it proves easiest to visualize any given secant ogive as the corresponding* tangent ogive of length λR, from which an axial length $\lambda_0 R$ has been truncated from the base end of the ogive. Thus, the ballistic length BL of the secant ogive may be expressed as $\mathsf{BL}/R = \lambda - \lambda_0$. The length R used for nondimensionalization is and remains the radius of the corresponding tangent ogive (not the secant radius!). Whereas a tangent ogive is wholly described in terms of two parameters f and R, the secant ogive requires one additional parameter, λ_0, characterizing the extent of truncation.

In the equations that follow, the term a is shorthand for

$$a = f - 1 \quad .$$

The term λ is the nondimensional length of the corresponding tangent ogive, given in terms of f by Eq. 1, as $\lambda = \sqrt{2f - 1}$. The term λ_0 represents the nondimensional length of the corresponding tangent ogive that is removed from the ogive base, so as to obtain the desired secant ogive. A depiction of this is shown in Fig. 3.

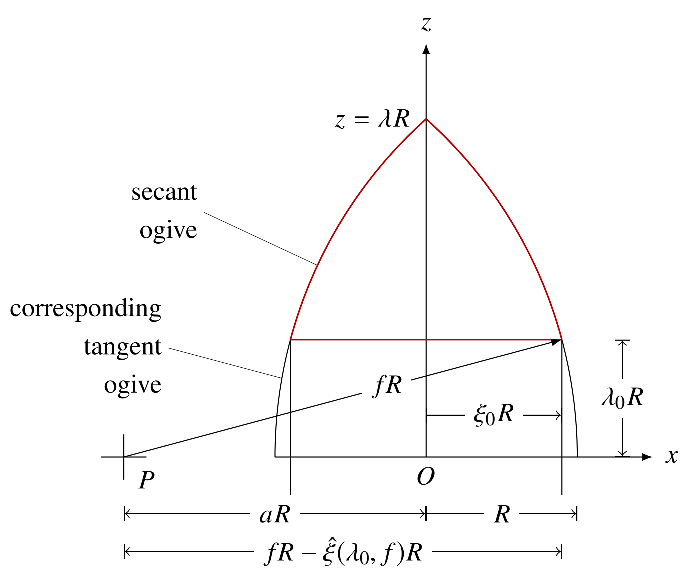

Fig. 3 Secant ogive seen as a truncation of the corresponding tangent ogive

*By corresponding, we mean a tangent ogive with the same ogival nondimensional radius of curvature as the secant ogive of interest. Thus, a secant ogive and its corresponding tangent ogive share the same value of f.

Defining the intermediate function $\mathbb{V}$ in terms of f and a dummy variable η, as

$$\mathbb{V}(\eta, f) = \pi\left[(f^2 + a^2)\eta - \frac{\eta^3}{3} - a\left(\eta\sqrt{f^2 - \eta^2} + f^2 \sin^{-1}\left(\frac{\eta}{f}\right)\right)\right] \quad , \tag{9}$$

the secant-ogive volume (V) may be obtained (in terms of f, R, and λ_0) as

$$\frac{V}{R^3} = \mathbb{V}(\lambda, f) - \mathbb{V}(\lambda_0, f) \quad . \tag{10}$$

Defining the intermediate function $\mathbb{Z}$ as

$$\mathbb{Z}(\eta, f) = \frac{\pi}{V/R^3}\left[(f^2 + a^2)\frac{\eta^2}{2} - \frac{\eta^4}{4} + \frac{2}{3}a(f^2 - \eta^2)^{3/2}\right] \quad , \tag{11}$$

the secant-ogive CG ($\bar{z}$), taken with respect to the secant-ogive base, is obtained as

$$\frac{\bar{z}}{R} = \mathbb{Z}(\lambda, f) - \mathbb{Z}(\lambda_0, f) - \lambda_0 \quad . \tag{12}$$

Defining the intermediate function $\mathbb{I}_{zz}$ as

$$\begin{aligned}\mathbb{I}_{zz}(\eta, f) = \frac{\pi}{2}\Bigg[& f^4\eta - \frac{2}{3}f^2\eta^3 + \frac{1}{5}\eta^5 + 6f^2a^2\eta - 2a^2\eta^3 + a^4\eta \\ & - a\left(\eta(f^2 - \eta^2)^{3/2} + \frac{3}{2}f^2\eta\sqrt{f^2 - \eta^2} + \frac{3}{2}f^4 \sin^{-1}\left(\frac{\eta}{f}\right)\right) \\ & - 2a^3\left(\eta\sqrt{f^2 - \eta^2} + f^2 \sin^{-1}\left(\frac{\eta}{f}\right)\right)\Bigg]\end{aligned} \quad , \tag{13}$$

the moment of inertia of a secant ogive about its axis may be obtained as

$$\frac{I_{zz}}{\rho R^5} = \mathbb{I}_{zz}(\lambda, f) - \mathbb{I}_{zz}(\lambda_0, f) \quad . \tag{14}$$

An alternate formulation may be obtained for I_{zz} using the method of shells to set up the integration. The appearance of the term $\hat{\xi}$ (appearing in Fig. 3) represents a radial quantity wholly expressible in terms of the independent variables as

$$\hat{\xi}(\eta, f) = f - \sqrt{f^2 - \eta^2} \quad .$$

The b_i constants are defined identically as found in Eq. 5. Defining the intermediate function $\mathbb{J}_{zz}$ as

$$\mathbb{J}_{zz}(\eta, f) = -\pi\left(\left(f^2\cos^{-1}\left(\frac{\eta}{f}\right) + (f - \hat{\xi})\eta\right)b_0 + 2\left(\frac{b_1}{3} + \frac{b_2\hat{\xi}}{4} + \frac{b_3\hat{\xi}^2}{5}\right)\eta^3\right) \quad , \tag{15}$$

the moment of inertia of a secant ogive about its axis may be alternately obtained as

$$\frac{I_{zz}}{\rho R^5} = \mathbb{J}_{zz}(\lambda, f) - \mathbb{J}_{zz}(\lambda_0, f) - \frac{\pi}{2}\lambda_0\xi_0^4 \quad , \tag{16}$$

where ξ_0 is the nondimensional radius of the secant ogive, R_{secant}/R, given mathematically as

$$\frac{R_{\text{secant}}}{R} = \xi_0 = 1 - \hat{\xi}(\lambda_0, f) = 1 - f + \sqrt{f^2 - \lambda_0^2} \quad . \tag{17}$$

Defining the intermediate function $\mathbb{I}_{xx}$ as

$$\begin{aligned}\mathbb{I}_{xx}(\eta, f) = \frac{\pi}{4}\Bigg(&\left[f^4\eta + \tfrac{2}{3}f^2\eta^3 - \tfrac{3}{5}\eta^5\right] \\ &+ a\left[\eta(f^2 - \eta^2)^{3/2} - \tfrac{5}{2}f^2\eta\sqrt{f^2 - \eta^2} - \tfrac{5}{2}f^4\sin^{-1}(\eta/f)\right] \\ &+ 6a^2\left[f^2\eta - \tfrac{1}{9}\eta^3\right] \\ &- 2a^3\left[\eta\sqrt{f^2 - \eta^2} + f^2\sin^{-1}(\eta/f)\right] + a^4\eta\Bigg)\end{aligned} \quad , \tag{18}$$

the moment of inertia about an axis lying in the base of the corresponding tangent ogive, perpendicular to the axis of symmetry, may be obtained as

$$\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{tangent-base}} = \mathbb{I}_{xx}(\lambda, f) - \mathbb{I}_{xx}(\lambda_0, f) \quad . \tag{19}$$

That same moment, taken with respect to the CG of the secant ogive, is

$$\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{cg}} = \mathbb{I}_{xx}(\lambda, f) - \mathbb{I}_{xx}(\lambda_0, f) - \frac{V}{R^3}\left(\frac{\bar{z}}{R} + \lambda_0\right)^2 \quad . \tag{20}$$

For all the moments given previously, Eq. 8 may be used to convert the results in terms of mass, rather than volume. Discussion of how the results of this section were obtained is described in Appendix C. Validation of these results has been performed by the author, but is not shown here. However, the code for doing so is provided in Appendix D.

One last point to be made regarding secant ogives is that the current derivations are fundamentally expressed in terms of the geometrical properties of the corresponding tangent ogive (f, R, and λ_0). In the practical usage actually employed by ogive designers, the specification is given in terms of the secant geometry, by way of the secant radius R_{secant} and two CRH numbers. The first of the two numbers describes the hypothetical tangent CRH that would produce the given secant-ballistic length. The second of the two numbers describes the CRH associated with the actual curvature of the ogive, normalized to the secant diameter.

Expressed as "A/B CRH", these terms may be derived in terms of the current (f, R, λ_0) nomenclature as

$$A = \frac{1 + \left((\lambda - \lambda_0) \dfrac{R}{R_{\text{secant}}} \right)^2}{4}$$

and

$$B = \frac{f}{2} \frac{R}{R_{\text{secant}}} \quad ,$$

where λ is given by Eq. 1 and R_{secant} is given by Eq. 17. For example, an $f = 6$ ogive with no truncation ($\lambda_0 = 0$) represents a 3/3 CRH tangent ogive. However, if 25% of the tangent length were lopped off ($\lambda_0/\lambda = 0.25$), the resulting secant ogive would have a reduced diameter, $R_{\text{secant}}/R = 0.94$ and a 1.99/3.18 CRH. These variations can be explored with the code provided in Appendix D.

6. Conclusions

The inertial properties for ballistic ogives have been analytically calculated in closed form. The calculations are included for tangent ogives, defined by two independent parameters f and R, as well as for secant ogives, whose geometry is defined by three independent parameters f, R, and λ_0.

The properties derived include the ogival volume (mass), the CG, and the various moments of inertia (of course, the ogive density must also be known if these properties are to be expressed in terms of mass, rather than volume). The value of these results is immediately apparent to anyone conducting rigid-body analyses of axisymmetric ballistic bodies. In the absence of such closed-form expressions, one must instead rely on numerical integration methods, which are less accurate and computationally costlier by orders of magnitude.

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Appendix A. Integration of Inertial Properties for Tangent Ogives

A.1 Ogive Volume

The ogive volume can be integrated using either the shell or disk method. The author has done both and confirmed (after some manipulation) co-equal results. Here, however, for simplicity, only the disk method is presented. The relevant variables are presented in Fig. A-1, with the side view of the circular integration disk shown in gray. By nondimensionalizing all the variables, the integration may proceed in η over the limits $\eta = 0$ to $\eta = \lambda$.

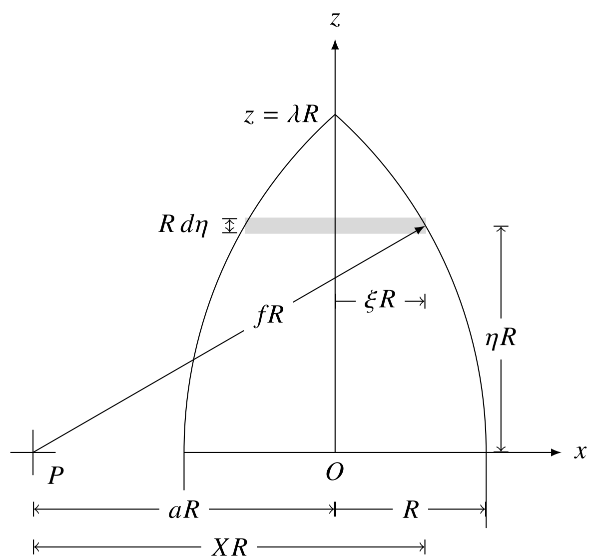

Fig. A-1 Disk method of ogive integration

We use the shorthand a to represent the fixed value $a = f - 1$. We introduce the variable of convenience, X, which can be expressed either in terms of the horizontal dimension (ξ) or alternately the vertical coordinate (η) of the disk as

$$X^2 = (a + \xi)^2 = f^2 - \eta^2 \quad . \tag{A-1}$$

As η varies from 0 to λ, X varies from f to a. The differential volume of the integration disk is

$$\frac{dV}{R^3} = \pi \xi^2 d\eta \quad . \tag{A-2}$$

The variable ξ can be eliminated in favor of X via substitution as

$$\frac{dV}{R^3} = \pi (X^2 - 2aX + a^2)\, d\eta \quad ,$$

followed by the elimination of X in favor of η:

$$\frac{dV}{R^3} = \pi\left(f^2 - \eta^2 - 2a\sqrt{f^2 - \eta^2} + a^2\right) d\eta \quad . \tag{A-3}$$

Since all the dimensional variables are positive, we retain the positive square root. The integration, which sets up as

$$\frac{V}{R^3} = \pi \int_0^{\lambda} \left(f^2 - \eta^2 - 2a\sqrt{f^2 - \eta^2} + a^2\right) d\eta \quad ,$$

may proceed, using standard integrals 7 and 200 in the CRC handbook[1]:

$$\frac{V}{R^3} = \pi\left[(f^2 + a^2)\eta - \frac{\eta^3}{3} - a\left(\eta\sqrt{f^2 - \eta^2} + f^2 \sin^{-1}\left(\frac{\eta}{f}\right)\right)\right]_{\eta=0}^{\eta=\lambda} \quad . \tag{A-4}$$

Noteworthy is the term $\sqrt{f^2 - \eta^2}$, which evaluates to a at the upper $\eta = \lambda$ limit, producing a $(-a^2\lambda)$ term that cancels with an earlier occurrence. The final result, eliminating a in favor of $f - 1$, becomes

$$\frac{V}{R^3} = \pi\left(\left(f^2 - \frac{1}{3}\lambda^2\right)\lambda - f^2(f-1)\sin^{-1}\left(\frac{\lambda}{f}\right)\right) \quad ,$$

given as Eq. 2 in the main body of this note.

A.2 Center of Gravity

The CG of the ogive, because of symmetry, lies along the axis of symmetry. Nonetheless, the z-distance from the base of the ogive is not immediately apparent. The well-known formula for determining the CG, in the context of the ogive geometry, is

$$\bar{z} = \frac{\int \eta R \, dV}{\int dV} = \frac{R}{V/R^3} \int \eta \frac{dV}{R^3} \quad .$$

This may be converted into a disk-based η integration by way of Eq. A-3:

$$\frac{\bar{z}}{R} = \frac{1}{V/R^3} \int_0^{\lambda} \pi\left(f^2 - \eta^2 - 2a\sqrt{f^2 - \eta^2} + a^2\right)\eta \, d\eta \quad .$$

We may again proceed, employing standard integrals 7 and 205 in the CRC handbook,[1]

[1]Beyer WH. CRC standard math tables. 26th ed. Boca Raton: CRC Press; 1981.

to obtain

$$\frac{\bar{z}}{R} = \frac{\pi}{V/R^3}\left[(f^2 + a^2)\frac{\eta^2}{2} - \frac{\eta^4}{4} - 2a\left(-\frac{1}{3}(f^2 - \eta^2)^{3/2}\right)\right]_{\eta=0}^{\lambda} \quad . \qquad \text{(A-5)}$$

Again noteworthy (by way of Eq. A-1) is the evaluation of $(f^2 - \eta^2)^{3/2}$. At the upper limit, it evaluates to a^3 and at the lower limit, to f^3. The final result, eliminating a in favor of $f - 1$, becomes

$$\frac{\bar{z}}{R} = \frac{\pi}{V/R^3}\left(-\frac{2}{3}(f-1)\big(f^3 - (f-1)^3\big) + \frac{1}{2}\big(f^2 + (f-1)^2\big)\lambda^2 - \frac{1}{4}\lambda^4\right) \quad ,$$

which is given as Eq. 3 in the main body of this note.

A.3 Moment of Inertia About the Symmetry Axis: I_{zz} (Disk Method)

Using the disk method to obtain the moment of inertia of the tangent ogive about the axis of symmetry is a simple exercise of summing the contributions of differential moment of inertia for each of the disks composing the ogive (see Fig. A-1). The moment of inertia of each disk about the axis of symmetry is given in any textbook reference[2] as

$$dI_{zz} = \frac{1}{2}r^2\,dm \quad .$$

This expression can be nondimensionalized, using Eq. A-2 and the notations of Fig. A-1, as

$$\frac{dI_{zz}}{\rho R^5} = \frac{1}{2}\xi^2\frac{dV}{R^3} = \frac{\pi}{2}\xi^4\,d\eta \quad .$$

Since ξ may be expressed as $\xi = X - a$ by way of Eq. A-1, we may expand the binomial power to get

$$\frac{dI_{zz}}{\rho R^5} = \frac{\pi}{2}\big(X^4 - 4aX^3 + 6a^2X^2 - 4a^3X + a^4\big)\,d\eta \quad .$$

The X term may be expressed in terms of η, by way of Eq. A-1, as

$$\frac{dI_{zz}}{\rho R^5} = \frac{\pi}{2}\left((f^2 - \eta^2)^2 - 4a(f^2 - \eta^2)^{3/2} + 6a^2(f^2 - \eta^2) - 4a^3\sqrt{f^2 - \eta^2} + a^4\right)d\eta \quad .$$

This expression needs to be integrated and evaluated between the limits $\eta = 0$ and

[2] Beer FP, Johnston Jr ER. Vector mechanics for engineers: Dynamics. 3rd ed. New York: McGraw Hill; 1977.

$\eta = \lambda$. We may employ standard integrals 7, 200, and 206 in the CRC handbook,[1] to obtain

$$\frac{I_{zz}}{\rho R^5} = \frac{\pi}{2}\Bigg[f^4\eta - \frac{2}{3}f^2\eta^3 + \frac{1}{5}\eta^5 + 6f^2a^2\eta - 2a^2\eta^3 + a^4\eta \\ - 4a \cdot \frac{1}{4}\left(\eta(f^2-\eta^2)^{3/2} + \frac{3}{2}f^2\eta\sqrt{f^2-\eta^2} + \frac{3}{2}f^4\sin^{-1}\left(\frac{\eta}{f}\right)\right) \\ - 4a^3 \cdot \frac{1}{2}\left(\eta\sqrt{f^2-\eta^2} + f^2\sin^{-1}\left(\frac{\eta}{f}\right)\right)\Bigg]_{\eta=0}^{\lambda} \quad . \tag{A-6}$$

We again make use of the fact that $\sqrt{f^2-\eta^2}$ evaluates to a at the upper limit $\eta = \lambda$. Terms may be collected and canceled to arrive at the final result

$$\frac{I_{zz}}{\rho R^5} = \frac{\pi}{2}\Bigg(\left(f^4 + \frac{9}{2}f(f-1)^2 - 2(f-1)^4\right)\lambda - \left(\frac{2}{3}f^2 + 2(f-1)^2\right)\lambda^3 + \frac{1}{5}\lambda^5 \\ - f^2(f-1)\left(\frac{3}{2}f^2 + 2(f-1)^2\right)\sin^{-1}\left(\frac{\lambda}{f}\right)\Bigg) \quad ,$$

given as Eq. 4 in the main body of this note.

A.4 Moment of Inertia About the Symmetry Axis: I_{zz} (Shell Method)

With the method of integration by shells, the axisymmetric domain is not sliced axially into thin disks, but rather constructed as a series of concentric hollow cylindrical shells, about the axis. The technique is illustrated in Fig. A-2, with the gray areas representing the radial section of a hollow cylindrical shell within the ogive. As such, the integration variable becomes ξ rather than η.

Such a cylindrical shell has volume based on its circumference and height:

$$\frac{dV}{R^3} = \eta(2\pi\xi)\,d\xi \quad .$$

The moment of inertia, about the axis of the shell, is merely

$$dI_{zz} = r^2\,dm \quad .$$

Nondimensionalized and using the notation of Fig. A-2, this expression becomes

$$\frac{dI_{zz}}{\rho R^5} = \xi^2\frac{dV}{R^3} = 2\pi\xi^3\eta\,d\xi \quad .$$

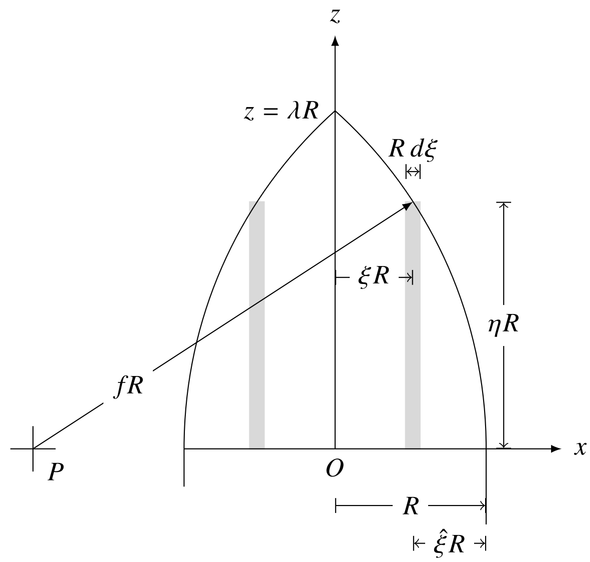

Fig. A-2 Shell method of ogive integration

We note, in light of Fig. A-2, the complementary variable, $\hat{\xi} = 1 - \xi$. Further, from the Pythagorean Theorem, we may establish that

$$(f - \hat{\xi})^2 + \eta^2 = f^2$$

We can use this to solve for η in terms of $\hat{\xi}$:

$$\eta = \sqrt{2f\hat{\xi} - \hat{\xi}^2} \quad .$$

We may substitute this for η and establish the differential for I_{zz} in terms of $\hat{\xi}$:

$$\frac{dI_{zz}}{\rho R^5} = -2\pi(1 - \hat{\xi})^3 \sqrt{2f\hat{\xi} - \hat{\xi}^2}\, d\hat{\xi} \quad ,$$

where $\hat{\xi}$ varies from 1 to 0 (as ξ varies from 0 to 1). Thus, we can eliminate the leading negative sign by instead varying $\hat{\xi}$ from 0 to 1.

The binomial can be expanded and the integration set up as

$$\frac{I_{zz}}{\rho R^5} = 2\pi \int_0^1 (1 - 3\hat{\xi} + 3\hat{\xi}^2 - \hat{\xi}^3)\sqrt{2f\hat{\xi} - \hat{\xi}^2}\, d\hat{\xi} \quad . \qquad \text{(A-7)}$$

These integrals appear as 264 and 266 in the CRC handbook.[1] However, in cases where the square root is multiplied by a power of $\hat{\xi}$, the integral produces a defined term and then reduces to an integral with a lower power of $\hat{\xi}$. Thus, there is something of an iteration, collecting terms at each step. The details of this integration are left to the reader. However, prior to evaluation, the integral may be expressed as

$$\frac{I_{zz}}{\rho R^5} = -\pi\left(\left(f^2\cos^{-1}\left(\frac{\eta}{f}\right) + (f-\hat{\xi})\eta\right)b_0 + 2\left(\frac{b_1\hat{\xi}^0}{3} + \frac{b_2\hat{\xi}^1}{4} + \frac{b_3\hat{\xi}^2}{5}\right)\eta^3\right)\Bigg|_{\hat{\xi}=0}^{1}, \quad \text{(A-8)}$$

where the b_i terms are constants. In Eq. A-8, the substitutions have already been made that $\sqrt{2f\hat{\xi} - \hat{\xi}^2} = \eta$ and $\sin^{-1}[(\hat{\xi} - f)/f] = -\cos^{-1}(\eta/f)$. Note that, during the subsequent limit evaluation, $\eta(\hat{\xi} = 0) = 0$ and $\eta(\hat{\xi} = 1) = \lambda$.

To better understand one of the previous substitutions, the evaluation of the integrals in Eq. A-7 produce terms of the form

$$\sin^{-1}\frac{\hat{\xi} - f}{f}\Bigg|_{\hat{\xi}=0}^{1} = \sin^{-1}\frac{1-f}{f} + \frac{\pi}{2} \quad .$$

Let the angle $\sin^{-1}[(1-f)/f]$ be denoted as α. Note that, since $f \geq 1$ is required, $\alpha \leq 0$ will result. Therefore, the value of $\alpha + \pi/2$ will represent the complement to α, call it β, as seen in Fig. A-3. The angle α may be recast as $-\cos^{-1}[\lambda/f]$.

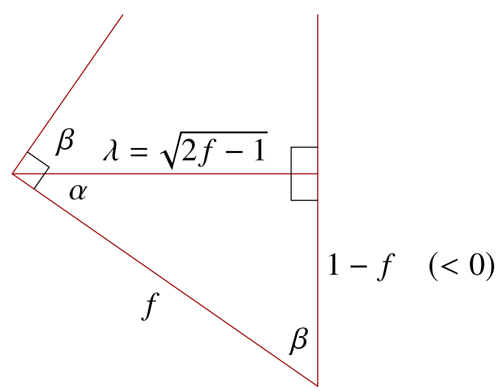

Fig. A-3 Recasting $\sin^{-1}[(1-f)/f]$ as $-\cos^{-1}(\lambda/f)$ and seeing β as the complement to α

This leads directly to the substitution

$$\sin^{-1}\frac{1-f}{f} + \frac{\pi}{2} = -\cos^{-1}\left(\frac{\lambda}{f}\right) + \frac{\pi}{2} = \sin^{-1}\left(\frac{\lambda}{f}\right) \quad .$$

The final result for the I_{zz} moment of inertia, by way of the shell method, is

$$\frac{I_{zz}}{\rho R^5} = \pi\left(\left(f^2 \sin^{-1}\left(\frac{\lambda}{f}\right) - (f-1)\lambda\right)b_0 - 2\left(\frac{b_1}{3} + \frac{b_2}{4} + \frac{b_3}{5}\right)\lambda^3\right) \quad ,$$

where

$$\begin{aligned} b_3 &= -1 \\ b_2 &= +3 + \tfrac{7}{5}\, b_3 f \\ b_1 &= -3 + \tfrac{5}{4}\, b_2 f \\ b_0 &= +1 + \tfrac{3}{3}\, b_1 f \end{aligned} \quad ,$$

which appears as Eq. 5 in the main body of this note.

A.5 Moment of Inertia About an Axis in the Ogive's Basal Plane: I_{xx}

We desire to use the disk method of integration to obtain the moment of inertia of the tangent ogive about an axis lying in the ogive's basal plane, which is perpendicular to the axis of symmetry. As before, we must sum the contributions of the differential moment of inertia for each of the disks composing the ogive (see Fig. A-1). The moment of inertia of each disk about an axis lying in the plane of the disk itself is given in any textbook reference[2] as

$$\left.\frac{dI_{xx}}{R^2}\right|_{\text{disk centered}} = \frac{1}{4}\xi^2\, dm \quad .$$

To obtain its contribution to the moment of inertia of the ogive, we must use the parallel-axis theorem to translate the reference axis from the center of the disk back to the base of the ogive:

$$\left.\frac{dI_{xx}}{R^2}\right|_{\text{base}} = \frac{1}{4}\xi^2\, dm + \eta^2\, dm \quad .$$

Nondimensionalize the expression, using the nomenclature of Fig. A-1:

$$\left.\frac{dI_{xx}}{\rho R^5}\right|_{\text{base}} = \frac{1}{4}(\xi^2 + 4\eta^2)\,\frac{dV}{R^3} = \frac{\pi}{4}(\xi^2 + 4\eta^2)\xi^2\, d\eta \quad .$$

Thus,

$$\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{base}} = \frac{\pi}{4}\int_0^{\lambda} (\xi^4 + 4\eta^2\xi^2)\, d\eta \quad . \tag{A-9}$$

With the help of Eq. A-1, we may express the individual terms as

$$\begin{aligned} \xi^4 &= X^4 - 4aX^3 + 6a^2X^2 - 4a^3X + a^4 \\ 4\eta^2\xi^2 &= 4(X^2 - 2aX + a^2)\eta^2 \end{aligned}$$

The powers of X may be expanded, in terms of η, using Eq. A-1:

$$\begin{aligned} X &= \sqrt{f^2 - \eta^2} \\ X^2 &= f^2 - \eta^2 \\ X^3 &= (f^2 - \eta^2)^{3/2} \\ X^4 &= (f^2 - \eta^2)^2 = f^4 - 2f^2\eta^2 + \eta^4 \end{aligned} \quad ,$$

which may be substituted into the prior expressions to yield the terms requiring integration

Term	CRC Integral[1]
$\xi^4\,d\eta = \left[f^4 - 2f^2\eta^2 + \eta^4\right] d\eta$	7
$-\,4a\left[(f^2 - \eta^2)^{3/2}\right] d\eta$	206
$+\,6a^2\left[f^2 - \eta^2\right] d\eta$	7
$-\,4a^3\left[\sqrt{f^2 - \eta^2}\right] d\eta$	200
$+\,a^4\,d\eta$	1
$4\eta^2\xi^2\,d\eta = 4\left[f^2\eta^2 - \eta^4\right] d\eta$	7
$-\,8a\left[\eta^2\sqrt{f^2 - \eta^2}\right] d\eta$	210
$+\,4a^2\left[\eta^2\right] d\eta$	7 .

Each integral (available in the cited reference[1]) may be substituted into Eq. A-9:

$$
\begin{aligned}
\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{base}} = \frac{\pi}{4}\Bigg(&\left[f^4\eta - \tfrac{2}{3}f^2\eta^3 + \tfrac{1}{5}\eta^5\right] \\
&- 4a\left[\tfrac{1}{4}\Big(\eta(f^2-\eta^2)^{3/2} + \tfrac{3}{2}f^2\eta\sqrt{f^2-\eta^2} + \tfrac{3}{2}f^4\sin^{-1}(\eta/f)\Big)\right] \\
&+ 6a^2\left[f^2\eta - \tfrac{1}{3}\eta^3\right] \\
&- 4a^3\left[\tfrac{1}{2}\Big(\eta\sqrt{f^2-\eta^2} + f^2\sin^{-1}(\eta/f)\Big)\right] \\
&+ a^4\eta \\
&+ 4\left[\tfrac{1}{3}f^2\eta^3 - \tfrac{1}{5}\eta^5\right] \\
&- 8a\left[-\tfrac{1}{4}\eta(f^2-\eta^2)^{3/2} + \tfrac{1}{8}f^2\Big(\eta\sqrt{f^2-\eta^2} + f^2\sin^{-1}(\eta/f)\Big)\right] \\
&+ 4a^2\left[\tfrac{1}{3}\eta^3\right]\Bigg)_{\eta=0}^{\lambda}
\end{aligned}
\quad .
$$

This expression can be somewhat simplified, even prior to evaluation at the limits:

$$
\begin{aligned}
\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{base}} = \frac{\pi}{4}\Bigg(&\left[f^4\eta + \tfrac{2}{3}f^2\eta^3 - \tfrac{3}{5}\eta^5\right] \\
&+ a\left[\eta(f^2-\eta^2)^{3/2} - \tfrac{5}{2}f^2\eta\sqrt{f^2-\eta^2} - \tfrac{5}{2}f^4\sin^{-1}(\eta/f)\right] \\
&+ 6a^2\left[f^2\eta - \tfrac{1}{9}\eta^3\right] \\
&- 2a^3\left[\eta\sqrt{f^2-\eta^2} + f^2\sin^{-1}(\eta/f)\right] + a^4\eta\Bigg)_{\eta=0}^{\lambda}
\end{aligned}
\quad . \tag{A-10}
$$

After evaluation between the limits $\eta = 0$ and $\eta = \lambda$, following much combination and cancellation of terms, and replacing a with $f-1$, the final result may be acquired as

$$
\begin{aligned}
\left.\frac{I_{xx}}{\rho R^5}\right|_{\text{base}} = \left.\frac{I_{yy}}{\rho R^5}\right|_{\text{base}} = \frac{\pi}{4}\Bigg(&f^2\Big(f^2 + \frac{7}{2}(f-1)^2\Big)\lambda + \frac{1}{15}\lambda^5 \\
&- f^2(f-1)\Big(\frac{5}{2}f^2 + 2(f-1)^2\Big)\sin^{-1}\Big(\frac{\lambda}{f}\Big)\Bigg)
\end{aligned}
\quad ,
$$

which is given as Eq. 6 in the main body of this note.

Appendix B. Tangent Ogive Validation Code

```
c     PROVIDES INERTIAL PROPERTIES FOR TANGENT OGIVES
c
c     The origin of this program's coordinate system is at the
c     center of the tangent-ogive base, z-axis along ogive
c     axis of symmetry.
c
c     Given bullet radius R and density rho,
c     Actual V    is       R**3 x that which is output
c     Actual mass is rho * R**3 x that which is computed from volume
c     Actual Izz  is rho * R**5 x that which is output
c     Actual Ixx  is rho * R**5 x that which is output
c     Actual Ixx/m      is R**2 x that which is output
c     Actual Izz/m      is R**2 x that which is output
c
      implicit none
      double precision f, Lm, h, r, V, Izz, PI, rbar, dr, crh, dh,
     &                 b0, b1, b2, b3, Ixx, hminus, hbar, rminus,
     &                 IhdV, cgz, Jzz
      integer n, i
      PARAMETER (n=1000)
      PI = 4.0d0*atan(1.d0)
      print *, 'Enter crh: '
      read (*,*) crh
      if (crh .lt. 0.5d0) then
        print *, 'crh must be 0.5 or greater.'
        stop
      end if
      f = crh * 2.d0
      print *, 'head-curvature radius    / tangent radius = ', f
      Lm = sqrt(2.d0*f - 1.d0)
      print *, 'Tangent Ballistic Length / tangent radius = ', Lm
      print *, 'NUMERICAL INTEGRATION (', n, ' STEPS)'
      V = 0.d0
      Ixx = 0.d0
      Izz = 0.d0
      IhdV = 0.d0
      dr = 1./float(n)
      do i = 1, n
        r      = (float(i) - 0.0d0)/float(n)
        rbar   = (float(i) - 0.5d0)/float(n)
        rminus = (float(i) - 1.0d0)/float(n)
        h      = sqrt(f**2 - (f-1 + r     )**2)
        hbar   = sqrt(f**2 - (f-1 + rbar  )**2)
        hminus = sqrt(f**2 - (f-1 + rminus)**2)
        dh = hminus - h
        V    = V    + hbar * 2.d0 * PI * rbar * dr! EITHER
c        V    = V    + PI * rbar**2 * dh!           OR
        Izz  = Izz  + hbar * 2.d0 * PI * rbar**3 * dr
        Ixx  = Ixx  + PI/4.d0*rbar**2 *
     &                (rbar**2 + 4.d0*hbar**2)*dh
        IhdV = IhdV + hbar * PI*rbar**2 * dh
      end do
      cgz = IhdV / V
      write(*,99) 'Volume = ', V
```

```fortran
      write(*,99) 'CGz    = ', cgz,   ' (wrt tangent-ogive base)'
      write(*,98) 'Izz    = ', Izz
      write(*,99) 'Izz/m    = ', Izz/V, ' (axis)'
      write(*,98) 'Ixx    = ', Ixx
      write(*,99) 'Ixx/m    = ', Ixx/V,
     &                        ' (about tangent-ogive base)'
      write(*,98) 'Ixx_0  = ', Ixx - V*cgz**2
      write(*,99) 'Ixx_0/m  = ', (Ixx/V - cgz**2),
     &                        ' (about tangent ogive CG)'
      print *, 'ANALYTICAL'
c     TANGENT-OGIVE VOLUME
      V = PI* ((f**2 - 1.d0/3.d0*Lm**2)*Lm - f**2*(f-1.d0)*asin(Lm/f))
c     INTEGERAL h dV AND THE CENTER OF GRAVITY
      IhdV = PI * (
     &         - 2.d0/3.d0*(f-1.d0)*(f**3 - (f-1.d0)**3)
     &         + 0.5d0*(f**2 + (f-1.d0)**2)*Lm**2
     &         - 0.25d0*Lm**4 )
      cgz = IhdV / V
c     MOMENT OF INERTIA ABOUT OGIVE AXIS OF SYMMETRY (DISKS)
      Izz = .5d0*PI *(
     & (f**4 + 4.5d0*f**2*(f-1.d0)**2 - 2.d0*(f-1.d0)**4)*Lm
     & -(2.d0/3.d0*f**2 + 2.d0*(f-1.d0)**2)*Lm**3 +Lm**5/5.d0
     & -f**2*(f-1.d0)*asin(Lm/f)*(1.5d0*f**2 + 2.d0*(f-1.d0)**2)
     & )
c     MOMENT OF INERTIA ABOUT OGIVE AXIS OF SYMMETRY (SHELLS)
      b3 = -1.d0
      b2 = +3.d0 + b3 * f * 7.d0/5.d0
      b1 = -3.d0 + b2 * f * 5.d0/4.d0
      b0 = +1.d0 + b1 * f * 3.d0/3.d0
      Jzz = PI*( (f**2 * asin(Lm/f) - (f - 1.d0) * Lm) * b0
     &       -2.0d0 * (b1/3.d0 + b2/4.d0 + b3/5.d0) * Lm**3 )
c     MOMENT OF INERTIA ABOUT CENTER OF TANGENT-OGIVE BASE
      Ixx = PI/4.d0 * (
     &  + f**2*(f**2 + 3.5d0*(f-1.d0)**2)*Lm + 1.d0/15.d0*Lm**5
     &  - f**2*(f-1.d0)*(2.5d0*f**2 + 2.d0*(f-1.d0)**2) * asin(Lm/f)
     &  )
      write(*,99) 'Volume = ', V
      write(*,99) 'CGz    = ', cgz, ' (wrt tangent-ogive base)'
      write(*,98) 'Izz    = ', Izz
      write(*,99) 'Izz/m    = ', Izz/V, ' (axis - disks)'
      write(*,98) 'Izz    = ', Jzz
      write(*,99) 'Izz/m    = ', Jzz/V, ' (axis - shells)'
      write(*,98) 'Ixx    = ', Ixx
      write(*,99) 'Ixx/m    = ', Ixx/V,
     &                        ' (about tangent-ogive base)'
      write(*,98) 'Ixx_0  = ', Ixx - V*cgz**2
      write(*,99) 'Ixx_0/m  = ', (Ixx/V - cgz**2),
     &                        ' (about tangent-ogive CG)'
   98 format(a,f8.5,'          ',$)
   99 format(a,f8.5,a)
      stop
      end
```

Appendix C. Extension of the Method to Secant Ogives

The comparable results for secant ogives are presented in Section 5 of this note. While the resulting expressions may be more complicated *vis-à-vis* tangent ogives, the approach itself is identical to that already presented in Appendix A, with a few minor changes. There is no mathematical impediment to pursuing this avenue. In fact, the validated *f77* code for doing so is presented in Appendix D.

For the disk-method integrations already described in Appendix A, it is only the limits of integration that need be changed for the secant-ogive case. While BL is the secant-ogive ballistic length, we retain $\lambda = \mathsf{BL}_{\tan}/R = \sqrt{2f-1}$ (Eq. 1) to describe the nondimensional ballistic length of the tangent ogive corresponding to the same ogival radius of curvature of the secant ogive under consideration. Define $\lambda_0 = (\mathsf{BL}_{\tan} - \mathsf{BL})/R$ to describe the nondimensional difference between the corresponding tangent and given secant-ogive ballistic lengths. The term λ_0 may be thought of as the nondimensional length truncated from the base of the corresponding tangent ogive to produce the secant ogive of interest.

To achieve the disk-method integrations for secant ogives, merely take the integrations (Eqs. A-4, A-5, A-6, and A-10) between the limits λ_0 and λ, rather than 0 to λ. The CG $\bar{z}$ will be with respect to the corresponding tangent-ogive base, unless shifted to the secant-ogive base by a distance $\lambda_0 R$.

Likewise, in the case of Eq. A-10, the "base" axis, about which the I_{xx} moment of inertia is calculated, is the base of the corresponding tangent ogive, not the secant ogive. Therefore, the parallel axis theorem must be employed (see Eq. 7) to shift the moment axis to the CG of the secant ogive. Note that, as $\lambda_0 \to 0$, the secant-ogive geometry approaches that of the corresponding tangent ogive.

For shell-method integration, two two alterations are required. As in the disk method, the lower limit of the $\hat{\xi}$ integration (Eq. A-8) needs to be replaced with $\hat{\xi}_0$, where the term $\hat{\xi}_0 = 1 - \xi_0 = f - \sqrt{f^2 - \lambda_0^2}$ represents a nondimensional radius difference between the secant and corresponding tangent ogives.

Second, the height of the cylindrical shell, given nondimensionally by the term η in the shell integration, must be decremented by an offset value of λ_0. This decrement will introduce an additional term to the moment-of-inertia integrand of Eq. A-7, corresponding to the I_{zz} moment of inertia of the cylinder falling outside the secant ogive, but inside the corresponding tangent ogive. That cylinder is located in the domain $z/R = [0, \lambda_0]$ and $r/R = [0, \xi_0]$ and the decremental term takes the form of $\Delta I_{zz}/\rho R^5 = -2\pi\lambda_0 \int_{\hat{\xi}_0}^{1} (1-\hat{\xi})^3 \, d\hat{\xi} = -\frac{1}{2}\pi\lambda_0\xi_0^4$. As in the case of λ_0, the term $\hat{\xi}_0 \to 0$ when the secant-ogive geometry approaches that of the corresponding tangent ogive.

Appendix D. Secant Ogive Validation Code

```
c     PROVIDES INERTIAL PROPERTIES FOR SECANT OGIVES
c
c     Think of the secant ogive as a tangent ogive that has had
c     a fraction of its length lopped off at the base.
c
c     The origin of this program's coordinate system is at the
c     center of the tangent-ogive base, z-axis along ogive
c     axis of symmetry.
c
c     Given bullet radius R and density rho,
c     Actual V    is       R**3 x that which is output
c     Actual mass is rho * R**3 x that which is computed from volume
c     Actual Izz  is rho * R**5 x that which is output
c     Actual Ixx  is rho * R**5 x that which is output
c     Actual Ixx/m      is R**2 x that which is output
c     Actual Izz/m      is R**2 x that which is output
c
      implicit none
      double precision f, Lm, h, r, V, Izz, PI, rbar, dr, crh, dh,
     &                 b0, b1, b2, b3, Ixx, hminus, hbar, rminus,
     &                 IhdV, cgz, L0, Lmfrac, Rsc, Jzz, crhtan,
     &          Vfunc, xbarfunc, Izzfuncdisk, Ixxfunc, Izzfuncshell
      integer n, i
      PARAMETER (n=100000)
      PI = 4.0d0*atan(1.d0)
      print *, 'Enter tangent crh AND ',
     &         'fraction of tangent-ogive length lopped off: '
      read (*,*) crhtan, Lmfrac
      if (crhtan .lt. 0.5d0) then
        print *, 'tangent crh must be 0.5 or greater.'
        stop
      end if
      f = crhtan * 2.d0
      print *, 'head-curvature radius    / tangent radius = ', f
      Lm = sqrt(2.d0*f - 1.d0)
      print *, 'Tangent Ballistic Length / tangent radius = ', Lm
      L0 = Lmfrac * Lm
      print *, 'Secant Ballistic Length  / tangent radius = ', Lm-L0
      print *, 'Lopped Length            / tangent radius = ', L0
      Rsc = 1.d0 - f + sqrt(f**2 - L0**2)
      print *, 'Secant-Ogive radius      / tangent radius = ', Rsc
      crh = crhtan / Rsc
      print *, 'head-curvature radius    / secant diam.   = ', crh
      write(*,'(/,f6.3,''/'',f6.3,''crh'',/)')
     &                  (1.d0 + ((Lm-L0)/Rsc)**2)/4.d0, crh
      print *, 'NUMERICAL INTEGRATION (', n, ' STEPS)'
      V = 0.d0
      Ixx = 0.d0
      Izz = 0.d0
      IhdV = 0.d0
      dr = Rsc/float(n)
      do i = 1, n
        r      = (float(i) - 0.0d0)/float(n) * Rsc
        rbar   = (float(i) - 0.5d0)/float(n) * Rsc
```

```
      rminus = (float(i) - 1.0d0)/float(n) * Rsc
      h      = sqrt(f**2 - (f-1 + r     )**2) - L0
      hbar   = sqrt(f**2 - (f-1 + rbar  )**2) - L0
      hminus = sqrt(f**2 - (f-1 + rminus)**2) - L0
      dh = hminus - h
      V    = V    + hbar * 2.d0 * PI * rbar * dr! EITHER
c       V    = V    + PI * rbar**2 * dh!           OR
      Izz  = Izz  + hbar * 2.d0 * PI * rbar**3 * dr
      Ixx  = Ixx  + PI/4.d0*rbar**2 *
     &                (rbar**2 + 4.d0*(L0 + hbar)**2)*dh
      IhdV = IhdV + (L0 + hbar) * PI*rbar**2 * dh
    end do
    cgz = IhdV / V
    write(*,99) 'Volume = ', V
    write(*,99) 'CGz    = ', cgz,   ' (wrt tangent-ogive base)'
    write(*,99) 'CGz    = ', cgz-L0, ' (wrt secant-ogive base)'
    write(*,98) 'Izz    = ', Izz
    write(*,99) 'Izz/m    = ', Izz/V, ' (axis)'
    write(*,98) 'Ixx    = ', Ixx
    write(*,99) 'Ixx/m    = ', Ixx/V,
   &                        ' (about tangent-ogive base)'
    write(*,98) 'Ixx_0  = ', Ixx - V*cgz**2
    write(*,99) 'Ixx_0/m  = ', (Ixx/V - cgz**2),
   &                        ' (about secant-ogive CG)'
    print *, 'ANALYTICAL'
c     TANGENT-OGIVE VOLUME
    V = PI * (Vfunc(Lm,f) - Vfunc(L0,f))
c     INTEGERAL h dV AND THE CENTER OF GRAVITY
    IhdV = PI * (xbarfunc(Lm,f) - xbarfunc(L0,f))
    cgz = IhdV / V - L0
c     MOMENT OF INERTIA ABOUT OGIVE AXIS OF SYMMETRY (DISKS)
    Izz = .5d0*PI * (Izzfuncdisk(Lm,f) - Izzfuncdisk(L0,f))
c     MOMENT OF INERTIA ABOUT OGIVE AXIS OF SYMMETRY (SHELLS)
    Jzz = -PI * (Izzfuncshell(Lm,f) - Izzfuncshell(L0,f)
   &      + 0.5d0*L0*Rsc**4 )
c     MOMENT OF INERTIA ABOUT CENTER OF TANGENT-OGIVE BASE
    Ixx = PI/4.d0 * (Ixxfunc(Lm,f) - Ixxfunc(L0,f))
    write(*,99) 'Volume = ', V
    write(*,99) 'CGz    = ', cgz+L0, ' (wrt tangent-ogive base)'
    write(*,99) 'CGz    = ', cgz, ' (wrt secant-ogive base)'
    write(*,98) 'Izz    = ', Izz
    write(*,99) 'Izz/m    = ', Izz/V, ' (axis - disks)'
    write(*,98) 'Izz    = ', Jzz
    write(*,99) 'Izz/m    = ', Jzz/V, ' (axis - shells)'
    write(*,98) 'Ixx    = ', Ixx
    write(*,99) 'Ixx/m    = ', Ixx/V,
   &                        ' (about tangent-ogive base)'
    write(*,98) 'Ixx_0  = ', Ixx - V*(cgz + L0)**2
    write(*,99) 'Ixx_0/m  = ', (Ixx/V - (cgz + L0)**2),
   &                        ' (about secant-ogive CG)'
 98 format(a,f8.5,'          ',$)
 99 format(a,f8.5,a)
    stop
    end
```

```
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
      double precision function Vfunc(h, f)
      double precision f, a, h, X2
      a = f - 1.d0
      X2 = f**2 - h**2
      Vfunc = (f**2 + a**2)*h - 1.d0/3.d0 * h**3 -
     &        a*(h * sqrt(X2) + f**2 * asin(h/f))
      return
      end
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
      double precision function xbarfunc(h, f)
      double precision f, a, h, X2
      a = f - 1.d0
      X2 = f**2 - h**2
      xbarfunc = (f**2 + a**2)/2.d0 * h**2 - 0.25 * h**4 +
     &           2.d0/3.d0 * a * X2**1.5d0
      return
      end
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
      double precision function Izzfuncdisk(h, f)
      double precision f, a, h, X2
      a = f - 1.d0
      X2 = f**2 - h**2
      Izzfuncdisk = f**4*h - 2.d0/3.d0*f**2*h**3 + 0.2d0*h**5 +
     &          6.d0*f**2*a**2*h - 2.d0*a**2*h**3 + a**4*h -
     &          a*(h*X2**1.5d0 + 1.5d0*f**2*h*sqrt(X2) +
     &             1.5d0*f**4*asin(h/f)) -
     &          2.d0*a**3*(h*sqrt(X2) + f**2*asin(h/f))
      return
      end
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
      double precision function Izzfuncshell(h, f)
      double precision f, xi, h, X2, b0, b1, b2, b3
      X2 = f**2 - h**2
      xi = f - sqrt(X2)
      b3 = -1.d0
      b2 = +3.d0 + b3 * f * 7.d0/5.d0
      b1 = -3.d0 + b2 * f * 5.d0/4.d0
      b0 = +1.d0 + b1 * f * 3.d0/3.d0
      Izzfuncshell = (f**2*acos(h/f) + (f-xi)*h) * b0 + 2.d0*(
     &               b1/3.d0 + b2*xi/4.d0 + b3*xi**2/5.d0)*h**3
      return
      end
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
      double precision function Ixxfunc(h, f)
      double precision f, a, h, X2
      a = f - 1.d0
      X2 = f**2 - h**2
      Ixxfunc = f**4*h + 2.d0/3.d0*f**2*h**3 - 0.6d0*h**5 +
     &          a*(h*X2**1.5d0 - 2.5d0*f**2*h*sqrt(X2) -
     &             2.5d0*f**4*asin(h/f))+
     &          6.d0*a**2*(f**2*h - 1.d0/9.d0*h**3) -
     &          2.d0*a**3*(h*sqrt(X2) + f**2*asin(h/f)) +
     &          a**4*h
```

```
      return
      end
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
```

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